Indian Statistical Institute, Bangalore

M. Math. First Year

First Semester - Analysis of Several Variables

Final Exam Date : November 09, 2015

Section I: Answer any four, each question carries 6 marks

1. Let $\mathbb{E}$ be an open set in $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}^m$ be differentiable at a point $x \in E$. Prove that components of f have partial derivatives at x. Is the converse true? Justify your answer.

2. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable and $f(tx) = tf(x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Can f be linear.

3. Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be an n-dimensional interval in $\mathbb{R}^n$ and $f\colon I \to \mathbb{R}$ be a continuous function. Prove that f is Riemann-integrable over I.

4. Evaluate $\int\int_D (3x - 2y) d(x, y)$ where D is parallelogram bounded by the lines $3x - 2y = 8$, $3x - 2y = -4$ and $2x + y = 1$, $2x + y = 3$.

5. Do flips and primitive maps on $\mathbb{R}^d$ have potential? Justify your answer.

6. Let E be an open set in $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}^n$ be continuous whose line integrals are independent of paths in S. Prove that there is a C^1-map $\phi\colon E \to \mathbb{R}$ such that $\nabla\phi = f$.

Section II: Answer any two, each question carries 13 marks

1. (a) Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at a point x and $f(x) = 0$. If $g\colon \mathbb{R}^n \to \mathbb{R}$ is continuous at x, prove that gf is differentiable at x.

   (b) Let E be an open set in $\mathbb{R}^n$ and $f\colon E \to \mathbb{R}^n$ be a C^1-map. Assume that $J_f(x) \neq 0$ for some $x \in E$. Prove that there is an open set $U \subset E$ containing x such that $f(U)$ is open and f restricted to U has a C^1-inverse (**Marks: 7**).

2. (a) Find the maximum volume of a rectangular box to be made from a cardboard of size $128m^2$ (**Marks: 8**).

   (b) Prove that $\int_{Q_n} x_1^{r_1} \cdots x_n^{r_n} dx = \frac{r_1! \cdots r_n!}{(n+r_1+\cdots+r_n)!}$ where Q_n is the standard simplex in $\mathbb{R}^n$ and each r_i is a non-negative integer.

3. (a) Let R be a rectangle with positively-oriented boundary Γ. If $u, v\colon R \to \mathbb{R}$ have continuous second order partial derivatives, prove that

$$\int_\Gamma u(x+y)dx + u(x+y)dy = \int\int_R (x+y)(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y})d(x,y)$$

and

$$\int_\Gamma (v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x})dx + (u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y})dy = 2\int\int_R (u\frac{\partial^2 v}{\partial x \partial y} - v\frac{\partial^2 u}{\partial x \partial y})d(x,y).$$

(b) Let T be an open connected set in $\mathbb{R}^2$ whose boundary is a piecewise smooth Jordan curve Γ with positive orientation and r is a function defined on an open set containing $T \cup \Gamma$ such that r has continuous second order partial derivatives and r is 1-1 on $T \cup \Gamma$. If P is a scalar-valued C^1-function defined on an open set containing $S = r(T \cup \Gamma)$, prove that $\int_{r(\Gamma)} P dx = \int\int_S \frac{\partial P}{\partial z} dz \wedge dx - \frac{\partial P}{\partial y} dx \wedge dy$ *(Marks: 7)*.